Partial Fraction Expansion

Examples are provided below for performing Partial Fraction Expansion (PFE) using the following methods:
1) by hand
2) using the TI-85 or TI-86 calculator
3) using the HP-48G or HP-48GX calculator
4) using the TI-89 or TI-92 calculator

Case 1: Functions with repeated linear roots

Consider the following example:

\[ F(s) = \frac{6s}{(s+1)(s+2)^2} \]

F(s) should be decomposed for Partial Fraction Expansion as follows:

\[ F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \]

A) Find A, B, and C by hand:

Using the residue method:

\[
A = \left. (s+1)F(s) \right|_{s=-1} = \left. \frac{6s}{(s+2)} \right|_{s=-1} = \frac{-6}{1^2} = -6
\]

\[
C = \left. (s+2)^2F(s) \right|_{s=-2} = \left. \frac{6s}{(s+1)} \right|_{s=-2} = \frac{-12}{-1} = 12
\]

\[
B = \left. \frac{d}{ds} (s+2)^2F(s) \right|_{s=-2} = \left. \frac{d}{ds} \left( \frac{6s}{(s+1)^2} \right) \right|_{s=-2} = \left. \frac{6}{(s+1)^2} \right|_{s=-2} = \frac{6}{(-1)^2} = 6
\]

so

\[ F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2} \quad \text{and} \quad f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t) \]

B) Find A, B, and C using the TI-85 or TI-86:

Use the TI-85 or TI-86 to determine A, B, and C as follows:

Run the program PARTIALF  (Select PROGRAM - NAMES - PARTIALF)
NUMBER OF DISTINCT LINEAR FACTORS  2
NUMBER OF DISTINCT QUADRATIC FACTORS  0
ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING
POWERS.

COEFFICIENTS OF LINEAR FACTOR NUMBER 1  1
1

ORDER OF LINEAR FACTOR NUMBER 1  1

COEFFICIENTS OF LINEAR FACTOR NUMBER 2  1
2

ORDER OF LINEAR FACTOR NUMBER 2  2

ENTER DEGREE OF NUMERATOR  6

ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS  0

The result is stored in LIST under the variable name CONST (Select LIST - NAMES - CONST)

\{ -6 6 12 \}

so \( F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2} \) and \( f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t) \)

C) Find A, B, and C using the HP-48G or HP-48GX:

\( F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^3} \)

Use the HP-48G or HP-48GX to determine A, B, and C as follows:

1. Load the information onto the stack as follows:

<table>
<thead>
<tr>
<th>Stack Contents</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>{6 0}</td>
<td>Coefficients in the numerator of F(s)</td>
</tr>
<tr>
<td>{1 1}</td>
<td>Coefficients for 1st factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 1st factor is raised</td>
</tr>
<tr>
<td>{1 2}</td>
<td>Coefficients for 2nd factor in denominator</td>
</tr>
<tr>
<td>2</td>
<td>Power to which the 2nd factor is raised</td>
</tr>
<tr>
<td>2</td>
<td>The total number of factors in the denominator</td>
</tr>
</tbody>
</table>

2. Run the program PARTIALF (located in the directory BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).

\{ -6 6 12 \}

so \( F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2} \) and \( f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t) \)
Case 2: Functions with complex roots

If a function $F(s)$ has a complex pole (i.e., a complex root in the denominator), it can be handled in two ways:
1) By keeping the complex roots in the form of a quadratic
2) By finding the complex roots and using complex numbers to evaluate the coefficients

**Example:** Both methods will be illustrated using the following example. Note that the quadratic terms has complex roots.

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)}$$

**Method 1: Quadratic factors in $F(s)$**

$F(s)$ should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17}$$

A) Find $A$, $B$, and $C$ by hand (for the quadratic factor method):
Combining the terms on the right with a common denominator and then equating numerators yields:

\[
A(s^2 + 2s + 17) + (Bs + C)(s + 1) = 5s^2 - 6s + 21
\]

Equating $s^2$ terms: $A + B = 5$
Equating $s$ terms: $2A + B + C = -6$
Equating constants: $17A + C = 21$

\[
\begin{align*}
A &= 2 \\
B &= 3 \\
C &= -13
\end{align*}
\]

so

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{2}{s + 1} + \frac{3s - 13}{s^2 + 2s + 17}$$

now manipulating the quadratic term into the form for decaying cosine and sine terms:

$$F(s) = \frac{2}{s + 1} + \frac{3(s + 1)}{(s + 1)^2 + 4^2} + \frac{-4(4)}{(s + 1)^2 + 4^2}$$

so

$$f(t) = e^{-t}[2 + 3\cos(4t) - 4\sin(4t)]u(t)$$

The two sinusoidal terms may be combined if desired using the following identity:

$$A\cos(wt) + B\sin(wt) = \sqrt{A^2 + B^2} \cos\left(wt - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

(or convert $(A, -B)$ to polar form)

so

$$f(t) = e^{-t}[2 + 5\cos(4t + 53.13^\circ)]u(t)$$
B) Find A, B, and C using the TI-85 or TI-86 (for the quadratic factor method)

F(s) should be decomposed for Partial Fraction Expansion as follows:

\[ F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17} \]

Use the TI-85 to determine A, B, and C as follows:

Run the program PARTIALF (Select PROGRAM - NAMES - PARTIALF)
(Calculator prompts are shown below in upper case ITALICS and user inputs are shown BOLD)

\[
\begin{align*}
\text{NUMBER OF DISTINCT LINEAR FACTORS} & : 1 \\
\text{NUMBER OF DISTINCT QUADRATIC FACTORS} & : 1 \\
\text{ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING POWERS.} \\
\text{COEFFICIENTS OF LINEAR FACTOR NUMBER 1} & : 1 \\
\text{ORDER OF LINEAR FACTOR NUMBER 1} & : 1 \\
\text{ENTER COEFFICIENTS AND ORDER OF QUADRATIC FACTORS BY DESCENDING POWERS.} \\
\text{COEFFICIENTS OF QUADRATIC FACTOR NUMBER 1} & : 1 \\
\text{ORDER OF QUADRATIC FACTOR NUMBER 1} & : 1 \\
\text{ENTER DEGREE OF NUMERATOR} & : 2 \\
\text{ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS} & : 5 \\
\end{align*}
\]

so \( F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{2}{s + 1} + \frac{3s - 13}{s^2 + 2s + 17} \)

so \( f(t) = e^t[2 + 3\cos(4t) - 4\sin(4t)]u(t) \)

or \( f(t) = e^t[2 + 5\cos(4t + 53.13^\circ)]u(t) \)

(see above for details on finding \( f(t) \) from \( F(s) \))

C) Find A, B, and C using the HP-48G or HP-48GX (for the quadratic factor method)

F(s) should be decomposed for Partial Fraction Expansion as follows:

\[ F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17} \]
Use the HP-48 to determine A, B, and C as follows:

1. Load the information onto the stack as follows:

<table>
<thead>
<tr>
<th>Stack Contents</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>{5 -6 21}</td>
<td>Coefficients in the numerator of F(s)</td>
</tr>
<tr>
<td>{1 1}</td>
<td>Coefficients for 1st factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 1st factor is raised</td>
</tr>
<tr>
<td>{1 2 17}</td>
<td>Coefficients for 2nd factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 2nd factor is raised</td>
</tr>
<tr>
<td>2</td>
<td>The total number of factors in the denominator</td>
</tr>
</tbody>
</table>

2. Run the program PARTIALF (located in the directory BOBM so use 2nd-HOME-VAR-BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).

\{2 3 -13\}

so

\[
F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{2}{s + 1} + \frac{3s - 13}{s^2 + 2s + 17}
\]

so

\[
f(t) = e^t[2 + 3\cos(4t) - 4\sin(4t)]u(t) \quad \text{or} \quad f(t) = e^t[2 + 5\cos(4t + 53.13^\circ)]u(t)
\]

(see above for details on finding f(t) from F(s))

Method 2: Complex roots in F(s)

Note that the roots of \((s^2 + 2s + 17)\) are \(s_1, s_2 = \alpha \pm j\omega = -1 \pm j4\)

so

\[
F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{5s^2 - 6s + 21}{(s + 1)(s + 1 - j4)(s + 1 + j4)}
\]

A) Find A, B, and C by hand (for the complex root method):

F(s) should be decomposed for Partial Fraction Expansion as follows:

\[
F(s) = \frac{A}{s + 1} + \frac{\overline{B}}{s + 1 - j4} + \frac{\overline{B}^*}{s + 1 + j4}
\]

where \(\overline{B}\) is a complex number

and \(\overline{B}^*\) is the conjugate of \(\overline{B}\).

The inverse transform of the two terms with complex roots will yield a single time-domain term of the form \(2\overline{B} = 2B/\theta = 2Be^{\alpha t}\cos(wt + \theta)\)

Using the Residue Theorem:

\[
A = (s + 1)F(s)\bigg|_{s=-1} = \left.\frac{5s^2 - 6s + 21}{(s^2 + 2s + 17)}\right|_{s=-1} = \frac{32}{16} = 2
\]
\[
\overline{B} = (s + 1 - j4)F(s)\bigg|_{s = -1 + j4} = \frac{5s^2 - 6s + 21}{(s + 1)(s + 1 + j4)}\bigg|_{s = -1 + j4} = \frac{5(-1,4)^2 - 6(-1,4) + 21}{(-1 + j4 + 1)(-1 + j4 + 1 + j4)} = 2.5 / 53.13^\circ
\]

It is not necessary to also find \( \overline{B}^* \), but doing so here illustrates the conjugate relationship.

\[
\overline{B}^* = (s + 1 + j4)F(s)\bigg|_{s = -1 - j4} = \frac{5s^2 - 6s + 21}{(s + 1)(s + 1 - j4)}\bigg|_{s = -1 - j4} = \frac{5(-1,-4)^2 - 6(-1,-4) + 21}{(-1 - j4 + 1)(-1 - j4 + 1 - j4)} = 2.5 / -53.13^\circ
\]

So, \( f(t) = 2e^{-t}u(t) + 2B/\theta = 2e^{-t}u(t) + 5/53.13^\circ \)

\[
\begin{align*}
\overline{f}(t) & = \left[2e^{-t} + 5e^{-t}\cos(4t + 53.13^\circ)\right]u(t)
\end{align*}
\]

This can be broken up into separate sine and cosine terms using

\[
\cos(\omega t + \theta) = \cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t)
\]

(or convert (R, j\theta) to (A, -B))

so \( \overline{f}(t) = \left[2e^{-t} + 5e^{-t}\left[\cos(53.13^\circ)\cos(4t) - \sin(53.13^\circ)\sin(4t)\right]\right]u(t) \)

\[
\begin{align*}
\overline{f}(t) & = \left[2e^{-t} + e^{-t}\left[4\cos(4t) - 3\sin(4t)\right]\right]u(t)
\end{align*}
\]

\textbf{B) Find A, B, and C using the TI-85 or TI-86 (for the complex root method)}

F\( (s) \) should be decomposed for Partial Fraction Expansion as follows:

\[
F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{\overline{B}}{s + 1 - j4} + \frac{\overline{B}^*}{s + 1 + j4}
\]

Use the TI-85 to determine \( A, \overline{B}, \) and \( \overline{B}^* \) as follows:

Run the program PARTIALF (Select PROGRAM - NAMES - PARTIALF)

(Calculator prompts are shown below in upper case \textit{ITALICS} and user inputs are shown \textbf{BOLD})

\begin{align*}
\text{NUMBER OF DISTINCT LINEAR FACTORS} & = 3 \\
\text{NUMBER OF DISTINCT QUADRATIC FACTORS} & = 0 \\
\text{ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING POWERS.} \\
\text{COEFFICIENTS OF LINEAR FACTOR NUMBER 1} & = 1 \\
\text{ORDER OF LINEAR FACTOR NUMBER 1} & = 1 \\
\text{COEFFICIENTS OF LINEAR FACTOR NUMBER 2} & = 1
\end{align*}
ORDER OF LINEAR FACTOR NUMBER 2  
1  
COEFFICIENTS OF LINEAR FACTOR NUMBER 3  
1  
ORDER OF LINEAR FACTOR NUMBER 3  
1  
ENTER DEGREE OF NUMERATOR  
2  
ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS  
5  
-6  
21  

The result is stored in LIST under the variable name CONST (Select LIST - NAMES - CONST)

{ (2.0 /0.00 ) (2.5 /53.13 ) (2.5 /-53.13 ) }  

so  \[ f(t) = \left[ 2e^t + 5e^t \cos(4t + 53.13) \right] u(t) \]  

or  \[ f(t) = \left[ 2e^t + e^t[4\cos(4t) - 3\sin(4t)] \right] u(t) \]  

(see above for details on finding \( f(t) \) from \( F(s) \))

C) Find A, B, and C using the HP-48G or HP-48GX (for the complex root method)

F(s) should be decomposed for Partial Fraction Expansion as follows:

\[ F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{B}{s + 1 - j4} + \frac{\overline{B}}{s + 1 + j4} \]

Use the HP-48 to determine \( A \), \( \overline{B} \), and \( \overline{B}^* \) as follows:

1. Load the information onto the stack as follows:

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>{5 -6 21}</td>
<td>Coefficients in the numerator of ( F(s) )</td>
</tr>
<tr>
<td>{1 1}</td>
<td>Coefficients for 1st factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 1st factor is raised</td>
</tr>
<tr>
<td>{1 (1, -4)}</td>
<td>Coefficients for 2nd factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 2nd factor is raised</td>
</tr>
<tr>
<td>{1 (1, 4)}</td>
<td>Coefficients for 3rd factor in denominator</td>
</tr>
<tr>
<td>1</td>
<td>Power to which the 3rd factor is raised</td>
</tr>
<tr>
<td>3</td>
<td>The total number of factors in the denominator</td>
</tr>
</tbody>
</table>

2. Run the program PARTIALF (located in the directory BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).

\{ (2,0) (2.5/-53.13) (2.5/+53.13) \}  

so  \[ f(t) = \left[ 2e^t + 5e^t \cos(4t + 53.13) \right] u(t) \]  

or  \[ f(t) = \left[ 2e^t + e^t[4\cos(4t) - 3\sin(4t)] \right] u(t) \]  

(see above for details on finding \( f(t) \) from \( F(s) \))
Transferring the Partial Fractions Expansion program between calculators

**TI-85 or TI-86**
1) Connect the two calculators with a cable
2) On the receiving calculator, select 2\textsuperscript{nd} -\textit{ LINK } - \textit{RECV}
3) On the sending calculator, select 2\textsuperscript{nd} - \textit{ LINK } - \textit{SEND } - \textit{PRGM}
   then either select the desired programs or press \textit{ALL+} to select all programs
   then select \textit{XMIT}

**HP-48G or HP-48GX**
1) Line up the arrows on the top each calculator corresponding to the infrared port
2) On the receiving calculator, select \textit{HOME} to insure that the calculator is in the HOME directory
   Then select \textit{I/O} and then \textit{GET FROM HP 48}
3) On the sending calculator, select \textit{HOME}
   Then select \textit{I/O} and then \textit{SEND TO HP 48}
   Then select \textit{EDIT } - \textit{ VAR } - \textit{ BOBM } - \textit{ ENTER} (this selects the entire BobMaynard directory,
   including all subprograms, for transfer)
   Then select \textit{SEND}

**Reference**: The Partial Fractions Expansion program for both the TI and HP calculators was
written by: Bob Maynard
Tidewater Community College Math Department
Phone: 822-7174
email: tcmaynr@tcc.edu
Partial Fractions Decomposition with the TI-89/TI-92

1) From the Algebra pull down menu (F2) select \textbf{3:expand(} either by using the cursor key to highlight the function then pressing enter, or entering 3 on the numeric keypad.

2) Enter the rational expression you wish to perform partial fraction decomposition on. CAUTION: be sure to use parentheses as needed to ensure proper grouping of terms in the numerator and denominator.

3) Close the \textbf{expand(} function with a right parenthesis.

4) Press enter.

\textbf{Example:}

To perform partial fraction decomposition of the expression

\[
\frac{s + 2}{s \cdot (s + 1)^3}
\]

1) Select the expand function \textbf{expand(}

2) Enter \((s + 2)/(s*(s + 1)^3)\)

\textbf{expand}((s + 2)/(s*(s + 1)^3))

3) Close the parentheses on the expand function \textbf{expand}((s + 2)/(s*(s + 1)^3))

4) Press Enter. The calculator will display the result

\[
\frac{-2}{s + 1} + \frac{-2}{(s + 1)^2} + \frac{-1}{(s + 1)^3} + \frac{2}{s}
\]