EGR 261
Signals and Systems
File: Partial Fractions

Partial Fraction Expansion

Examples are provided below for performing Partial Fraction Expansion (PFE) using the following methods:
1) by hand
2) using the TI-85 or TI-86 calculator
3) using the HP-48G or HP-48GX calculator
4) using the TI-89 or TI-92 calculator

Case 1: Functions with repeated linear roots

Consider the following example:

$$F(s) = \frac{6s}{(s+1)(s+2)^2}$$

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

A) Find A, B, and C by hand:

Using the residue method:

$$A = (s+1)F(s)\Big|_{s=-1} = \frac{6s}{(s+2)}\Bigg|_{s=-1} = \frac{-6}{1^2} = -6$$

$$C = (s+2)^2F(s)\Big|_{s=-2} = \frac{6s}{(s+1)}\Bigg|_{s=-2} = \frac{-12}{-1} = 12$$

$$B = \frac{d}{ds}(s+2)^2F(s)\Big|_{s=-2} = \frac{d}{ds}\left[\frac{6s}{(s+1)}\right]\Bigg|_{s=-2} = \frac{6}{(s+1)^2}\Bigg|_{s=-2} = \frac{6}{(-1)^2} = 6$$

so $F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2}$ and $\boxed{f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t)}$

B) Find A, B, and C using the TI-85 or TI-86:

$$F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Use the TI-85 or TI-86 to determine A, B, and C as follows:

Run the program PARTIALF (Select PROGRAM - NAMES - PARTIALF)

(Calculator prompts are shown below in upper case *ITALICS* and user inputs are shown **BOLD**)

NUMBER OF DISTINCT LINEAR FACTORS **2**
NUMBER OF DISTINCT QUADRATIC FACTORS **0**
ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING POWERS.
COEFFICIENTS OF LINEAR FACTOR NUMBER 1 **1**
1
ORDER OF LINEAR FACTOR NUMBER 2 **1**
COEFFICIENTS OF LINEAR FACTOR NUMBER 2 **1**
2
ORDER OF LINEAR FACTOR NUMBER 2 **2**
ENTER DEGREE OF NUMERATOR **1**
ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS **6**
0
The result is stored in LIST under the variable name CONST (Select LIST - NAMES - CONST)

{ -6 6 12 }

so $F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2}$ and $\boxed{f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t)}$

C) <u>Find A, B, and C using the HP-48G or HP-48GX</u>:

$$F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Use the HP-48G or HP-48GX to determine A, B, and C as follows:

1. Load the information onto the stack as follows:

Stack Contents	**Comments**
{6 0}	Coefficients in the numerator of F(s)
{1 1}	Coefficients for 1st factor in denominator
1	Power to which the 1st factor is raised
{1 2}	Coefficients for 2nd factor in denominator
2	Power to which the 2nd factor is raised
2	The total number of factors in the denominator

2. Run the program PARTIALF (located in the directory BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).
   {-6 6 12}

so $F(s) = \frac{6s}{(s+1)(s+2)^2} = \frac{-6}{s+1} + \frac{6}{s+2} + \frac{12}{(s+2)^2}$ and $\boxed{f(t) = [-6e^{-t} + (6 + 12t)e^{-2t}]u(t)}$

Case 2: Functions with complex roots

If a function F(s) has a complex pole (i.e., a complex root in the denominator), it can be handled in two ways:

1) By keeping the complex roots in the form of a quadratic
2) By finding the complex roots and using complex numbers to evaluate the coefficients

Example: Both methods will be illustrated using the following example. Note that the quadratic terms has complex roots.

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)}$$

Method 1: Quadratic factors in F(s)

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17}$$

A) Find A, B, and C by hand (for the quadratic factor method):

Combining the terms on the right with a common denominator and then equating numerators yields:

$$A(s^2 + 2s + 17) + (Bs + C)(s + 1) = 5s^2 - 6s + 21$$

$$\left.\begin{array}{ll} \text{Equating } s^2 \text{ terms:} & A + B = 5 \\ \text{Equating } s \text{ terms:} & 2A + B + C = -6 \\ \text{Equating constants:} & 17A + C = 21 \end{array}\right\} \quad \text{yields} \quad \left\{\begin{array}{l} A = 2 \\ B = 3 \\ C = -13 \end{array}\right.$$

$$\text{so} \quad F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{2}{s + 1} + \frac{3s - 13}{s^2 + 2s + 17}$$

now manipulating the quadratic term into the form for decaying cosine and sine terms:

$$F(s) = \frac{2}{s + 1} + \frac{3(s + 1)}{(s + 1)^2 + 4^2} + \frac{-4(4)}{(s + 1)^2 + 4^2}$$

$$\text{so} \quad \boxed{f(t) = e^{-t}\left[2 + 3\cos(4t) - 4\sin(4t)\right]u(t)}$$

The two sinusoidal terms may be combined if desired using the following identity:

$$A\cos(wt) + B\sin(wt) = \sqrt{A^2 + B^2}\,\cos\left(wt - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

(or convert (A, -B) to polar form)

$$\text{so} \quad \boxed{f(t) = e^{-t}\left[2 + 5\cos(4t + 53.13^\circ)\right]u(t)}$$

B) <u>Find A, B, and C using the TI-85 or TI-86 (for the quadratic factor method)</u>

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17}$$

Use the TI-85 to determine A, B, and C as follows:

Run the program PARTIALF (Select PROGRAM - NAMES - PARTIALF)
(Calculator prompts are shown below in upper case *ITALICS* and user inputs are shown **BOLD**)

NUMBER OF DISTINCT LINEAR FACTORS **1**
NUMBER OF DISTINCT QUADRATIC FACTORS **1**
ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING POWERS.
COEFFICIENTS OF LINEAR FACTOR NUMBER 1 **1**
1
ORDER OF LINEAR FACTOR NUMBER 1 **1**
ENTER COEFFICIENTS AND ORDER OF QUADRATIC FACTORS BY DESCENDING POWERS.
COEFFICIENTS OF QUADRATIC FACTOR NUMBER 1 **1**
2
17
ORDER OF QUADRATIC FACTOR NUMBER 1 **1**
ENTER DEGREE OF NUMERATOR **2**
ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS **5**
-6
21

The result is stored in LIST under the variable name CONST (Select LIST - NAMES - CONST)

{2.0000 3.0000 -13.0000}

so $$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{2}{s + 1} + \frac{3s - 13}{s^2 + 2s + 17}$$

so $\boxed{f(t) = e^{-t}\left[2 + 3\cos(4t) - 4\sin(4t)\right]u(t)}$ or $\boxed{f(t) = e^{-t}\left[2 + 5\cos(4t + 53.13^\circ)\right]u(t)}$

(see above for details on finding f(t) from F(s))

C) <u>Find A, B, and C using the HP-48G or HP-48GX (for the quadratic factor method)</u>

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 2s + 17}$$

Use the HP-48 to determine A, B, and C as follows:

1. Load the information onto the stack as follows:

Stack Contents	**Comments**
{5 -6 21}	Coefficients in the numerator of F(s)
{1 1}	Coefficients for 1st factor in denominator
1	Power to which the 1st factor is raised
{1 2 17}	Coefficients for 2nd factor in denominator
1	Power to which the 2nd factor is raised
2	The total number of factors in the denominator

3. Run the program PARTIALF (located in the directory BOBM so use 2nd-HOME-VAR-BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).
   {2 3 -13}

$$\text{so } F(s) = \frac{5s^2 - 6s + 21}{(s+1)(s^2+2s+17)} = \frac{2}{s+1} + \frac{3s-13}{s^2+2s+17}$$

$$\text{so } \boxed{f(t) = e^{-t}\left[2 + 3\cos(4t) - 4\sin(4t)\right]u(t)} \quad \text{or} \quad \boxed{f(t) = e^{-t}\left[2 + 5\cos(4t + 53.13^\circ)\right]u(t)}$$

(see above for details on finding f(t) from F(s))

<u>Method 2: Complex roots in F(s)</u>

Note that the roots of $(s^2 + 2s + 17)$ are $s_1, s_2 = \alpha \pm jw = -1 \pm j4$

$$\text{so } F(s) = \frac{5s^2 - 6s + 21}{(s+1)(s^2+2s+17)} = \frac{5s^2 - 6s + 21}{(s+1)(s+1-j4)(s+1+j4)}$$

A) <u>Find A, B, and C by hand (for the complex root method)</u>:

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{A}{s+1} + \frac{\overline{\mathbf{B}}}{s+1-j4} + \frac{\overline{\mathbf{B}}^*}{s+1+j4}$$

where $\overline{\mathbf{B}}$ is a complex number and $\overline{\mathbf{B}}^*$ is the conjugate of $\overline{\mathbf{B}}$.

The inverse transform of the two terms with complex roots will yield a single time-domain term of the form $2\overline{\mathbf{B}} = 2B\underline{/\theta} = 2Be^{\alpha t}\cos(wt + \theta)$

Using the Residue Theorem:

$$A = (s+1)F(s)\Big|_{s=-1} = \left.\frac{5s^2 - 6s + 21}{(s^2+2s+17)}\right|_{s=-1} = \frac{32}{16} = 2$$

$$\bar{\mathbf{B}} = (s + 1 - j4)F(s)\Big|_{s=-1+j4} = \left.\frac{5s^2 - 6s + 21}{(s + 1)(s + 1 + j4)}\right|_{s=-1+j4}$$

$$= \frac{5(-1,4)^2 - 6(-1,4) + 21}{(-1 + j4 + 1)(-1 + j4 + 1 + j4)} = 2.5 \underline{/53.13°}$$

It is not necessary to also find $\bar{\mathbf{B}}^*$, but doing so here illustrates the conjugate relationship.

$$\bar{\mathbf{B}}^* = (s + 1 + j4)F(s)\Big|_{s=-1-j4} = \left.\frac{5s^2 - 6s + 21}{(s + 1)(s + 1 - j4)}\right|_{s=-1-j4}$$

$$= \frac{5(-1,-4)^2 - 6(-1,-4) + 21}{(-1 - j4 + 1)(-1 - j4 + 1 - j4)} = 2.5 \underline{/-53.13°}$$

So, $f(t) = 2e^{-t}u(t) + 2B\underline{/\theta} = 2e^{-t}u(t) + 5\underline{/53.13°}$

$$\boxed{f(t) = \left[2e^{-t} + 5e^{-t}\cos(4t + 53.13°)\right]u(t)}$$

This can be broken up into separate sine and cosine terms using

$\cos(wt + \theta) = \cos(\theta)\cos(wt) - \sin(\theta)\sin(wt)$

(or convert (R, $\underline{/\theta}$) to (A, - B))

so $f(t) = \left[2e^{-t} + 5e^{-t}\left[\cos(53.13°)\cos(4t) - \sin(53.13°)\sin(4t)\right]\right]u(t)$

$$\boxed{f(t) = \left[2e^{-t} + e^{-t}\left[4\cos(4t) - 3\sin(4t)\right]\right]u(t)}$$

B) <u>Find A, B, and C using the TI-85 or TI-86 (for the complex root method)</u>

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{\bar{\mathbf{B}}}{s + 1 - j4} + \frac{\bar{\mathbf{B}}^*}{s + 1 + j4}$$

Use the TI-85 to determine A, $\bar{\mathbf{B}}$, and $\bar{\mathbf{B}}^*$ as follows:

Run the program PARTIALF (Select PROGRAM - NAMES - PARTIALF)
(Calculator prompts are shown below in upper case *ITALICS* and user inputs are shown **BOLD**)

NUMBER OF DISTINCT LINEAR FACTORS **3**
NUMBER OF DISTINCT QUADRATIC FACTORS **0**
ENTER COEFFICIENTS AND ORDER OF EACH LINEAR FACTOR BY DESCENDING POWERS.
COEFFICIENTS OF LINEAR FACTOR NUMBER 1 **1**
1
ORDER OF LINEAR FACTOR NUMBER 1 **1**

COEFFICIENTS OF LINEAR FACTOR NUMBER 2 **1**

	(1,-4)
ORDER OF LINEAR FACTOR NUMBER 2	**1**
COEFFICIENTS OF LINEAR FACTOR NUMBER 3	**1**
	(1,4)
ORDER OF LINEAR FACTOR NUMBER 3	**1**
ENTER DEGREE OF NUMERATOR	**2**
ENTER COEFFICIENTS OF NUMERATOR BY DESCENDING POWERS	**5**
	-6
	21

The result is stored in LIST under the variable name CONST (Select LIST - NAMES - CONST)

{ (2.0 $\underline{/0.00}$) (2.5 $\underline{/53.13}$) (2.5 $\underline{/-53.13}$) }

so $\boxed{f(t) = \left[2e^{-t} + 5e^{-t}\cos(4t + 53.13°)\right]u(t)}$ or $\boxed{f(t) = \left[2e^{-t} + e^{-t}\left[4\cos(4t) - 3\sin(4t)\right]\right]u(t)}$

(see above for details on finding f(t) from F(s))

C) <u>Find A, B, and C using the HP-48G or HP-48GX (for the complex root method)</u>

F(s) should be decomposed for Partial Fraction Expansion as follows:

$$F(s) = \frac{5s^2 - 6s + 21}{(s + 1)(s^2 + 2s + 17)} = \frac{A}{s + 1} + \frac{\overline{\mathbf{B}}}{s + 1 - j4} + \frac{\overline{\mathbf{B}}^*}{s + 1 + j4}$$

Use the HP-48 to determine A, $\overline{\mathbf{B}}$, and $\overline{\mathbf{B}}^*$ as follows:

1. Load the information onto the stack as follows:

Stack Contents	**Comments**
{5 -6 21}	Coefficients in the numerator of F(s)
{1 1}	Coefficients for 1st factor in denominator
1	Power to which the 1st factor is raised
{1 (1, -4)}	Coefficients for 2nd factor in denominator
1	Power to which the 2nd factor is raised
{1 (1, 4)}	Coefficients for 3rd factor in denominator
1	Power to which the 3rd factor is raised
3	The total number of factors in the denominator

2. Run the program PARTIALF (located in the directory BOBM).

3. The results will now appear on the stack (in the same order as the factors were entered).
{(2,0) (2.5$\underline{/-53.13}$) (2.5$\underline{/+53.13}$) }

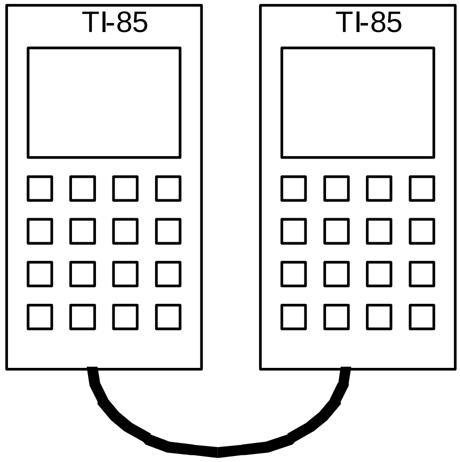
so $\boxed{f(t) = \left[2e^{-t} + 5e^{-t}\cos(4t + 53.13°)\right]u(t)}$ or $\boxed{f(t) = \left[2e^{-t} + e^{-t}\left[4\cos(4t) - 3\sin(4t)\right]\right]u(t)}$

(see above for details on finding f(t) from F(s))

Transferring the Partial Fractions Expansion program between calculators

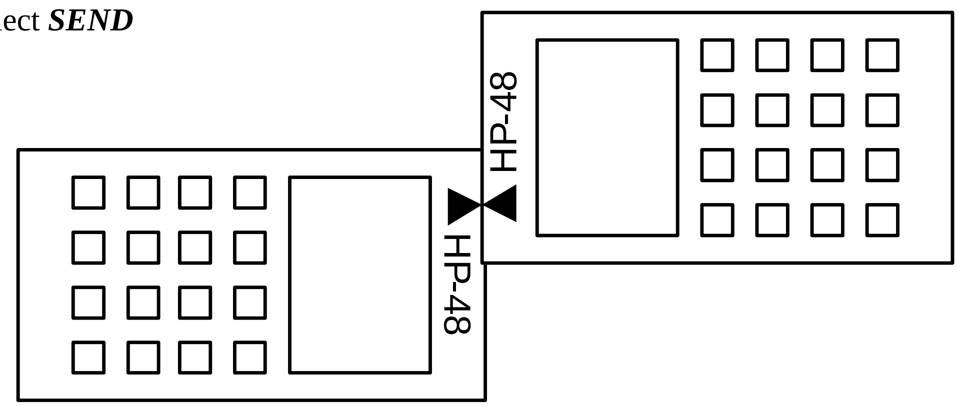
TI-85 or TI-86

1) Connect the two calculators with a cable
2) On the receiving calculator, select ***2nd - LINK - RECV***
3) On the sending calculator, select
   2nd - LINK - SEND - PRGM
   then either select the desired programs or press ***ALL+*** to select all programs
   then select ***XMIT***



HP-48G or HP-48GX

1) Line up the arrows on the top each calculator corresponding to the infared port
2) On the receiving calculator, select ***HOME*** to insure that the calculator is in the HOME directory
   Then select ***I/O*** and then ***GET FROM HP 48***
3) On the sending calculator, select HOME
   Then select ***I/O*** and then ***SEND TO HP 48***
   Then select ***EDIT - VAR - BOBM - ENTER*** (this selects the entire BobMaynard directory, including all subprograms, for transfer)
   Then select ***SEND***



Reference: The Partial Fractions Expansion program for both the TI and HP calculators was written by: Bob Maynard
Tidewater Community College Math Department
Phone: 822-7174
email: tcmaynr@tcc.edu

Partial Fractions Decomposition with the TI-89/TI-92

1) From the **Algebra** pull down menu (**F2**) select **3:expand(** either by using the cursor key to highlight the function then pressing enter, or entering **3** on the numeric keypad.
2) Enter the rational expression you wish to perform partial fraction decomposition on. CAUTION: be sure to use parentheses as needed to ensure proper grouping of terms in the numerator and denominator.
3) Close the **expand(** function with a right parenthesis.
4) Press enter.

Example:

To perform partial fraction decomposition of the expression

$$\frac{s+2}{s\cdot(s+1)^3}$$

1) Select the expand function
   expand(
2) Enter (s + 2)/(s*(s + 1)^3)
   expand((s + 2)/(s*(s + 1)^3)
3) Close the parentheses on the expand function
   expand((s + 2)/(s*(s + 1)^3))
4) Press Enter. The calculator will display the result

$$\frac{-2}{s+1} + \frac{-2}{(s+1)^2} + \frac{-1}{(s+1)^3} + \frac{2}{s}$$